ON THE INFINITE-DIMENSIONAL HIDDEN SYMMETRIES. III. q_R–CONFORMAL SYMMETRIES AT $q_R \to \infty$ AND BEREZIN-KARASEV-MASLOV ASYMPTOTIC QUANTIZATION OF $C^\infty(S^1)$

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The relations between the infinite dimensional geometry of q_R–conformal symmetries at $q_R \to \infty$, Berezin quantization of the Lobachevskiĭ plane and Karasev-Maslov asymptotic quantization are explicated. Various aspects of the "approximate" representation theory are discussed.

This short paper being the continuation of the previous parts [1] belongs to the series of articles supplemental to [2], and also lies in lines of the general ideology exposed in the review [3]. The main purpose of the activity, which has its origin and motivation presumably in the author's researches [4] on the experimentally mathematical aspects of interactively controlled systems (i.e. the controlled systems, in which the control is coupled with unknown or uncompletely known uncontrolled feedbacks) and applications, is to explicate the essentially *infinite-dimensional* aspects of the hidden symmetries, which appear in the representation theory of the finite dimensional Lie algebras and related algebraic structures. The present series is organized as a sequence of topics, which illustrate this basic idea on the simple and tame examples without superfluous difficulties and details as well as in the series [2] but from a bit more geometric point of view.

Topic Six: q_R–conformal symmetries at $q_R \to \infty$ and Berezin-Karasev-Maslov asymptotic quantization on $C^\infty(S^1)$

The main result of this topic, which may be regarded as an illustration to the book [5], is the following theorem.

Theorem. *Let us realize the Lobachevskiĭ-Berezin C^*–algebra [6-8] by the bounded operators in the unitarizable Verma module V_h over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ [8]. Then the lifting of the projective $\mathcal{HS}$–pseudorepresentation of the group $\mathrm{Diff}_+(S^1)$ in V_h [1:Topic 1] to its action in the Lobachevskiĭ-Berezin algebra realizes an asymptotic Karasev-Maslov-type quantization [5] (see also [9]) of $C^\infty(S^1)$ [supplied with the natural action of the group $\mathrm{Diff}_+(S^1)$] at $h \to \frac{1}{2}$.*

Let us comment the statement of the theorem.

In the Poincare realization of the Lobachevskii plane (the realization in the unit disk) the Lobachevskii metric may be written as $w = q_R^{-1}\, dzd\bar{z}/(1-|z|^2)^2$; and one can construct the C^*–algebra (Lobachevskiĭ-Berezin algebra) [6-8], which may be considered as a quantization of such metric [6], namely, let us consider two variables t and t^*, which obey the following commutation relations: $[tt^*, t^*t] = 0$, $[t, t^*] = q_R(1-tt^*)(1-t^*t)$ (or in an equivalent form $[ss^*, s^*s] = 0$, $[s, s^*] = (1-q_R ss^*)(1-q_R s^*s)$, where $s = (q_R)^{-1/2}t$); one may realize such variables by bounded operators in the Verma module over $\mathfrak{sl}(2,\mathbb{C})$ of the weight $h = \frac{q_R^{-1}+1}{2}$ [8]; if such Verma module is realized in polynomials of one complex variable z and the action of $\mathfrak{sl}(2,\mathbb{C})$ has the form $L_{-1} = z$, $L_0 = z\partial_z + h$, $L_1 = z(\partial_z)^2 + 2h\partial_z$, then the variables t and t^* are represented by tensor operators $D = \partial_z$ and $F = z/(z\partial_z + 2h)$. These operators are bounded if $q_R > 0$ and therefore one can construct a Banach algebra generated by them and obeying the prescribed commutation relations; the structure of C^*–algebra is rather obvious: an involution $*$ is defined on generators in a natural way, because the corresponding tensor operators are conjugate to each other.

The "classical" case corresponds to the limit $q_R \to 0$. However, it is interesting to consider another limit transition as $q_R \to \infty$. In this case the Lobachevskiĭ-Berezin algebra is reduced to the algebra $C^\infty(S^1)$ and the variable t^* is identified with t^{-1}. However, the algebra $C^\infty(S^1)$ possesses the group $\mathrm{Diff}_+(S^1)$ of the orientation preserving diffeomorphisms of a circle as a group of symmetries. This property is "weakly" conserved for finite q_R. Namely, the group $\mathrm{Diff}_+(S^1)$ admits projective $\mathcal{HS}$–pseudorepresentations in the Verma modules V_h over $\mathfrak{sl}(2,\mathbb{C})$ [1]; $\mathcal{HS}$–pseudorepresentation means the representation up to the Hilbert-Schmidt operators; the prefix "pseudo" is motivated by the analogy with the pseudodifferential calculus [10]. The infinitesimal versions of the projective $\mathcal{HS}$–pseudorepresentations of the group $\mathrm{Diff}_+(S^1)$, the $\mathcal{HS}$–projective representations of its Lie algebra $\mathrm{Vect}(S^1)$, the algebra of vector fields on a circle, were considered in [2:Topic 10]. The generators of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ in these $\mathcal{HS}$–projective representations are defined by the q_R–conformal symmetries [8] (the tensor operators of spin 2) and have the following form

$$L_k = (\xi + (k+1)h)\partial_z^k \quad (k \geq 0), \qquad L_{-k} = z^k \frac{\xi + (k+1)h}{(\xi + 2h)\ldots(\xi + 2h + k - 1)} \quad (k \geq 1),$$

where $\xi = z\partial_z$.

However, it is rather interesting to investigate the asymptotics of the $\mathcal{HS}$–pseudorepresentations of $\mathrm{Diff}_+(S^1)$ as $q_R \to \infty$ or $h \to \frac{1}{2}$. First of all, let us lift the projective $\mathcal{HS}$–pseudorepresentation of $\mathrm{Diff}_+(S^1)$ in the Verma module V_h to the Lobachevskiĭ-Berezin algebra as well as the corresponding $\mathcal{HS}$–projective representation of $\mathrm{Vect}(S^1)$. The analysis of the explicit formulas for the generators of the Lobachevskiĭ-Berezin algebra and q_R–conformal symmetries allows to state that the lifting of the $\mathcal{HS}$–projective representation of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ realizes a representation $mod\,\hbar$ ($\hbar = 2h-1 = q_R^{-1}$) of this Lie algebra in sense of [5,9], moreover, this representation is one by derivatives of the Lobachevskiĭ-Berezin algebra $mod\,\hbar$. The theorem is just the global version of this statement.

Note that $C^\infty(S^1)$ is not supplied by the structure of the Poisson algebra so our case slightly differs from one considered by M.V.Karasev and V.P.Maslov, who asymptotically quantize Poisson manifolds. However, the formal difference is subtle and immaterial.

The theorem may be combined with the results of [1:Topic 2].

Remark 1. The composed representations of the Witt isotopic pair [1:Topic 2] in the Verma modules V_h over the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ may be lifted to its representation $mod\,\hbar$ ($\hbar = 2h-1$) in the Lobachevskiĭ-Berezin algebra.

It is interesting to "globalize" this representation $mod\,\hbar$ of the Witt isotopic pair.

The theorem may be interpreted in terms of the nonlinear geometric algebra [11]. Recall that the projective $\mathcal{HS}$–pseudorepresentations of the group $G = \mathrm{Diff}_+(S^1)$ in the Verma modules V_h supply G with by the noncanonical groupuscular structures $\mathcal{G}_h$ [1:Topic 1], which are not odular ones.

Remark 2. The noncanonical groupuscular structures $\mathcal{G}_h$ on $G = \mathrm{Diff}_+(S^1)$ define the asymptotically canonical groupuscular structure $mod\,\hbar$ ($\hbar = 2h-1$).

Note that groupuscular structures (as well as general loopuscular structures) naturally appear in the formalism of the nonlinear Poisson brackets [5] and differential geometry [11] so their asymptotic considerations are of interest. Such considerations should combine the nonlinear geometric algebra [11] with the asymptotic algebraic and differential geometries [12].

Questions:
(1) What groupuscular structure on $G = \mathrm{Diff}_+(S^1)$ appears from the family $\mathcal{G}_h$ of noncanonical groupuscular structures $mod\,\hbar^2$ ($\hbar = 2h-1$)?
(2) Does the family of projective $\mathcal{HS}$–pseudorepresentations of the group $\mathrm{Diff}_+(S^1)$, which constitute the asymptotic projective representation $mod\,\hbar$ of $\mathrm{Diff}_+(S^1)$, define any hypergroup deformation [13;5:App.II] of this group? (3) Does the asymptotic representation of the Lie algebra $\mathrm{Vect}(S^1)$ admit an extension to the asymptotic representation of the Lie algebra $\mathrm{DOP}_{[\cdot,\cdot]}(S^1)$ of all differential operators on the circle?

Let us formulate some **conclusions**: (1) the relations between q_R–conformal symmetries at $q_R \to \infty$, the Berezin quantization of the Lobachevskiĭ plane and the Karasev-Maslov asymptotic quantization were explicated, (2) two approaches to the "approximate" representation theory, namely, one of the $\mathcal{HS}$–pseudorepresentations and one of the asymptotic representations $mod\,\hbar$ appeared as being closely and nontrivially connected in the considered case.

References

[1] Juriev D., On the infinite-dimensional hidden symmetries. I,II. E-prints: funct-an/9612004, funct-an/9701009.

[2] Juriev D., Topics in hidden symmetries. I-V. E-prints: hep-th/9405050, q-alg/9610026, q-alg/9611003, q-alg/9611019, funct-an/9611003.

[3] Juriev D.V., An excursus into the inverse problem of representation theory [in Russian]. Report RCMPI-95/04 (1995) [e-version: mp_arc/96-477].

[4] Juriev D.V., Octonions and binocular mobilevision [in Russian]. Fundam.Prikl.Matem., to appear; Belavkin-Kolokoltsov watch-dog effects in interactively controlled stochactic dynamical videosystems [in Russian]. Teor.Matem.Fiz. 106(2) (1996) 333-352 [English transl.: Theor.Math.Phys. 106 (1996) 276-290]; On the description of a class of physical interactive information systems [in Russian]. Report RCMPI/96-05 (1996) [e-version: mp_arc/96-459]; Droems: experimental mathematics, informatics and infinite dimensional geometry [in Russian]. Report RCMPI/96-05^{+} (1996).

[5] Karasev M.V., Maslov V.P., Nonlinear Poisson brackets: geometry and quantization. Amer.Math.Soc., Providence, RI, 1991.

[6] Berezin F.A., Quantization in complex symmetric spaces [in Russian]. Izvestiya AN SSSR. Ser.matem. 39(2) (1975) 363-402.

[7] Klimek S., Lesniewski A., Quantum Riemann surfaces. I. Commun.Math.Phys. 146 (1992) 103-122.

[8] Juriev D., Complex projective geometry and quantum projective field theory [in Russian]. Teor.Matem.Fiz. 101(3) (1994) 331-348 [English transl.: Theor.Math.Phys. 101 (1994) 1387-1403];

[9] Karasev M.V., Maslov V.P., Asymptotic and geometric quantization [in Russian]. Uspekhi Matem.Nauk 39(6) (1984) 115-173.

[10] Taylor M., Pseudo-differential operators. B., 1974; Duistermaat J., Fourier integral operators. N.Y., 1973; Treves F., Introduction to pseudo-differential and Fourier integral operators. N.Y., 1980.

[11] Sabinin L.V., Differential geometry and quasigroups [in Russian]. In "Current problems of geometry. To the 60th Anniversary of Acad.Yu.G.Reshetnyak". Trans.Inst.Math. Siberian Branch Soviet Acad.Sci., Novosibirsk, 1984, v.14, pp.208-221; On the nonlinear geometric algebra [in Russian]. In "Webs and quasigroups". Kalinin [Tver'], 1988, pp.32-37; Mikheev P.O., Sabinin L.V., Quasigroups and differential geometry. In "Quasigroups and loops. Theory and applications". Berlin, Heldermann Verlag, 1990, P.357-430.

[12] Dubnov V.L., Maslov V.P., Nazaĭkinskiĭ V.E., The complex Lagrangian germ and the canonical operator. Russian J.Math.Phys. 3 (1995) 141-180.

[13] Litvinov G.L., Hypergroups and hypergroup algebras [in Russian]. Current Probl.Math., Fundam.Directions, V.26, Moscow, VINITI, 1985, P.57-106; Levitan B.M., Theory of the generalized shift operators [in Russian]. Moscow, Nauka, 1973; Gurevich D.I., Generalized shift operators on Lie groups [in Russian]. Izvestiya AN Armyanskoi SSR. 18(4) (1983) 305-317.